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BIFURCATION OF PERIODIC SOLUTIONS OF A SYSTEM CLOSE TO A LYAPUNOV SYSTEM†

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Bifurcation of 2π-periodic solutions (2π-ps) of a system of second-order differential equations close to a Lyapunov system is investigated. The case of principal resonance, when an eigenfrequency of the linear oscillations of the unperturbed system is close to the frequency of the perturbing impulse, is considered. It is shown that, at certain values of the problem parameters, bifurcation of the 2π-ps that are generated from an equilibrium position, occurs. A constructive method is proposed for finding the bifurcation curve, as well as 2π-ps on it. The examples considered are bifurcation of 2π-ps in the problem of the oscillations of a mathematical pendulum with a horizontally vibrating suspension point, and in the problem of the planar oscillations of an artificial satellite in a weakly elliptical orbit. The bifurcation curves for these examples are constructed and the corresponding 2π-ps are found.

1. ON THE EXISTENCE OF PERIODIC SOLUTIONS ON THE BIFURCATION CURVE

Consider a system of second-order differential equations

$$dx/dt = -\omega y + X(x, y) + \mu F_1(x, y, t, \mu)$$
$$dy/dt = \omega x + Y(x, y) + \mu F_2(x, y, t, \mu) \tag{1.1}$$

The right-hand sides of system (1.1) are analytic functions of the variables x and y in some sufficiently small neighbourhood of the origin $x = y = 0$, such that the expansions of $X(x, y)$ and $Y(x, y)$ in the convergent series of powers of x and y begin with terms of at least second degree

$$X(x,y)=\sum_{k=2}^{\infty}\sum_{i+j=k}a_{ij}x^iy^j,\ Y(x,y)=\sum_{k=2}^{\infty}\sum_{i+j=k}b_{ij}x^iy^j \tag{1.2}$$

ω is a positive constant and μ is the small parameter of the problem When $\mu = 0$ system (1.1) is a Lyapunov system. The perturbing functions $F_i(x, y, t, \mu)$ $(i = 1, 2)$ are analytic in μ and 2π-periodic in t; their Fourier expansions at $x = y = \mu = 0$ are given by the formulae

$$F_i(0,0,t,0)=A_{i0}+\sum_{m=1}^{\infty}(A_{im}\cos mt+B_{im}\sin mt) \tag{1.3}$$

Suppose system (1.1) has a principal resonance [1], that is, ω is close to an integer n and, in addition, at least one of the quantities

$$D_1=\frac{1}{2}(A_{1n}+B_{2n}),\ D_2=\frac{1}{2}(A_{2n}-B_{1n}) \tag{1.4}$$

does not vanish. Further, we put

$$\omega = n + \delta \tag{1.5}$$

where δ is a small quantity.

The problem of the existence of 2π-periodic solutions (2π-ps) of system (1.1) generated from an equilibrium position of the unperturbed ($\mu = 0$) system has been investigated before [1, 2]. It has been shown [2] that the plane of the parameters μ, δ may be divided into two subdomains, in one of which system (1.1) has one 2π-ps and in the other three 2π-ps of the indicated form. The curve separating the two subdomains is known as the bifurcation curve of the solutions.

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In this paper we will consider the analytical construction of the bifurcation curve and 2π-ps on it as convergent series in fractional powers of μ.

Following the well-known approach [1, 2], we will seek initial values β_1 and β_2 of the variables x and y that satisfy the following 2π-periodicity conditions for solutions of system (1.1)

$$x(2\pi, \beta_1, \beta_2, \mu) = \beta_1, y(2\pi, \beta_1, \beta_2, \mu) = \beta_2 \tag{1.6}$$

Any sufficiently small β_1 and β_2 define when $\mu = 0$ a periodic solution with period

$$T = 2\pi\omega^{-1}[1 + h_{2l}(\beta_1^2 + \beta_2^2)^l + \ldots] \tag{1.7}$$

where the unwritten terms are small to a higher order than $O(\beta_i^{2l})$ ($i = 1, 2$) [3].

The required periodic solutions are analytic functions of β_1, β_2 and μ and may be expressed as

$$\begin{aligned} x(t, \beta_1, \beta_2, \mu) &= x(t, \beta_1, \beta_2, 0) + \mu[c_1(t) + \Phi_1(t, \beta_1, \beta_2, \mu)] \\ y(t, \beta_1, \beta_2, \mu) &= y(t, \beta_1, \beta_2, 0) + \mu[c_2(t) + \Phi_2(t, \beta_1, \beta_2, \mu)] \end{aligned} \tag{1.8}$$

where the functions Φ_1 are analytic in β_1, β_2 and μ and vanish when $\mu = \beta_1 = \beta_2 = 0$. Then the 2π-periodicity conditions (1.6) for x and y can be written in the form

$$\begin{aligned} x(2\pi, \beta_1, \beta_2, 0) + \mu[c_1(2\pi) + \Phi_1(2\pi, \beta_1, \beta_2, \mu)] &= \beta_1 \\ y(2\pi, \beta_1, \beta_2, 0) + \mu[c_2(2\pi) + \Phi_2(2\pi, \beta_1, \beta_2, \mu)] &= \beta_2 \end{aligned} \tag{1.9}$$

Substituting (1.8) into (1.1) and integrating the resulting equations for $c_i(t)$ with zero initial conditions, we obtain

$$c_i(2\pi) = 2\pi D_i + O(\delta) \tag{1.10}$$

The numbers D_i are determined from formulae (1.4). Expression (1.7) may be transformed, taking (1.5) into consideration, as follows [1]:

$$2\pi = nT + \delta\frac{2\pi}{n} - 2\pi h_{2l}(\beta_1^2 + \beta_2^2)^l + O(\beta^{2l+2}) + O(\delta^2) + O(\delta\beta^{2l}) \tag{1.11}$$

where $O(\beta^{2l+2})$ and $O(\delta\beta^{2l})$ are terms whose order relative to β_1 and β_2 is at least 21 + 2 and 21, respectively. It follows from Eqs (1.1) that

$$\begin{aligned} \left.\frac{dx}{dt}\right|_{t=nT,\mu=0} &= -n\beta_2 + O(\beta^2) + O(\delta\beta) \\ \left.\frac{dy}{dt}\right|_{t=nT,\mu=0} &= n\beta_1 + O(\beta^2) + O(\delta\beta) \end{aligned} \tag{1.12}$$

Expanding the left-hand sides of (1.6) in series in the neighbourhood of $t = nT$ and $\mu = 0$, and taking (1.11) and (1.12) into consideration, we rewrite the periodicity conditions (1.9) in the form

$$\begin{aligned} nh_{2l}\beta_2(\beta_1^2 + \beta_2^2)^l - \delta\beta_2 + \mu D_1 + \psi_1 &= 0 \\ nh_{2l}\beta_1(\beta_1^2 + \beta_2^2)^l - \delta\beta_1 - \mu D_2 + \psi_2 &= 0 \end{aligned} \tag{1.13}$$

where ψ_i are functions containing terms of orders no less than $O(\beta^{2l+2})$, $O(\delta^2\beta)$, $O(\delta\beta^2)$, $O(\delta\mu)$, $O(\mu\beta)$, $O(\mu^2)$.

Next, following the well-known approach of [2], we will confine our attention to the case in which the quantities $\delta\beta_i$ are of the same order as μ. It then follows from conditions (1.13) that $\beta_i = O(\mu^{1/(2l+1)})$, that is, $\delta = O(\mu^{2l/(2l+1)})$. Setting $\varepsilon = \mu^{1/(2l+1)}$, $\delta = \alpha\varepsilon^{2l}$, $\beta_i = b_i\varepsilon$ (the quantities α and b_i are of the order of unity), we deduce from the 2π-periodicity conditions (1.13) that

$$\begin{aligned} f_1 &\equiv nh_{2l}b_2(b_1^2 + b_2^2)^l - \alpha b_2 + D_1 + O(\varepsilon) = 0 \\ f_2 &\equiv nh_{2l}b_1(b_1^2 + b_2^2)^l - \alpha b_1 - D_2 + O(\varepsilon) = 0 \end{aligned} \tag{1.14}$$

System (1.14) yields the equation

$$b_1 f_1 - b_2 f_2 \equiv b_1 D_1 + b_2 D_2 + O(\varepsilon) = 0 \tag{1.15}$$

We will now assume, to fix our ideas, that $D_2 \neq 0$. Then, solving Eq. (1.15) for b_2 and substituting the result into the second equation of (1.14), we have

$$g(b_1, \alpha, \varepsilon) \equiv f_2(b_1, b_2(b_1, \varepsilon), \alpha, \varepsilon) \equiv b_1^{2l+1} + (2l+1)pb_1 + 2lq + O(\varepsilon) = 0 \tag{1.16}$$

$$p = -\frac{\alpha r}{(2l+1)nh_{2l}}, \quad q = -\frac{D_2 r}{2\ln h_{2l}}, \quad r = \left(\frac{D_2^2}{D_1^2 + D_2^2}\right)^l$$

Analysis of Eq. (1.16) shows [2] that when $\varepsilon = 0$ a real solution of Eq. (1.16) always exist, which depends continuously on the parameter α, which we denote by $b_1 = b_1^{(1)}$, with the property that $\partial g/\partial b \neq 0$ when $b_1 = b_1^{(1)}$, $\varepsilon = 0$. This means that for any values of α and sufficiently small ε, Eq. (1.16) has at least one real solution, which is expressible as a convergent series in powers of ε. This solution is unique if the following inequality holds

$$h_{2l}(\alpha - \alpha^{\text{cr}}) < 0 \tag{1.17}$$

$$\alpha^{\text{cr}} = (2l+1)nh_{2l}[(D_1^2 + D_2^2)^l / (2\ln h_{2l})^{2l}]^{1/(2l+1)}$$

But if inequality (1.17) holds with the inverse sign, then for sufficiently small ε two further real solutions of this form exist [2]. When $\varepsilon = 0$ and $\alpha = \alpha^{\text{cr}}$, Eq. (1.16) has exactly two real solutions. These parameter values correspond to a branch point of the solutions, that is, a point at which a new real solution of Eq. (1.16) is "born".

We will now investigate the bifurcation of solutions at small but non-zero ε. A necessary condition for the birth of a new real solution of Eq. (1.16) is that $\partial g/\partial b = 0$. Thus, in order to find a value of α corresponding to a branch point we must solve the following system of equations in α and b_i

$$g(b_1, \alpha, \varepsilon) = 0, \quad \varphi(b_1, \alpha, \varepsilon) \equiv \partial g / \partial b_1 (b_1, \alpha, \varepsilon) = 0 \tag{1.18}$$

When $\varepsilon = 0$ the only real solution of system (1.18) is α^{cr}, $b_1^{\text{cr}} = -(2l+1)/(2l\alpha)^{\text{cr}}$. The Jacobian

$$\partial(g, \varphi)/\partial(b, \alpha)|_{\alpha=\alpha^{\text{cr}}, b_1 = b_1^{\text{cr}}, \varepsilon=0} = (2l+1)2l(b_1^{\text{cr}})^{2l} r / (nh_{2l})$$

does not vanish, and therefore, for sufficiently small ε, system (1.18) has a unique solution which may be expressed as a convergent series in powers of ε

$$b_1^* = b_1^{\text{cr}} + \varepsilon b_{11} + \ldots, \quad \alpha^* = \alpha^{\text{cr}} + \varepsilon \alpha_1 + \ldots \tag{1.19}$$

In order to show that at values of the parameter α corresponding to (1.19) the solutions of Eq. (1.16) do indeed bifurcate, we expand the left-hand side of the equation in a Taylor series about b_1^* and α^*

$$B(b_1 - b_1^*)^2 - A(\alpha - \alpha^*) + \ldots = 0 \tag{1.20}$$

$$B = (2l+1)l(b_1^*)^{2l-1} + O(\varepsilon), \quad A = b_1^* r / (nh_{2l}) + O(\varepsilon)$$

The quantities b_1^* and α^* depend on ε; the dots denote terms of orders $O((b_1 - b_1^*)^3)$, $O((\alpha - \alpha^*)^2)$, $O((\alpha - \alpha^*)(b_1 - b_1))$, $O(\varepsilon)$. Equation (1.20) is known as the bifurcation equation [4]. If $h_{2l}(\alpha - \alpha^*) > 0$ and the quantity $|\alpha - \alpha^*|$ is sufficiently small, Eq. (1.20) has exactly two real solutions with respect to b_1, which may be expressed as convergent series in powers of $\surd(|\alpha - \alpha^*|)$. The coefficients of these series are analytic functions of ε. These solutions merge, becoming b_1^*, as $\alpha \to \alpha^*$. But if $h_{2l}(\alpha - \alpha^*) < 0$, Eq. (1.20) has no real solutions with respect to b_1. Thus at the values of α defined by formula (1.19) the solutions of Eq. (1.16) do indeed bifurcate, and hence do so the 2π-ps of system (1.1). Taking (1.5) into account, we can derive from (1.19) the following expression for ω in terms of the parameter μ

$$\omega = n + \mu^{2l/(2l+1)}\alpha^{\text{cr}} + O(\mu) \tag{1.21}$$

which defines the bifurcation curve of 2π-ps of system (1.1), dividing the domain of the parameters ω and μ into two subdomains: in one of them one 2π-ps exists while in the other there are three 2π-ps of system (1.1), which turn into the trivial solution as $\mu \to 0$. On the curve itself, system (1.1) has two 2π-ps of the indicated form.

2. A METHOD FOR FINDING 2π-ps ON THE BIFURCATION CURVE

We will consider a method for the practical construction of the bifurcation curve and of 2π-ps on the curve. For simplicity we will confine the discussion to the case when $l = 1$.

As shown previously, the bifurcation curve and 2π-ps on it should be sought as series in powers of the small parameter $\varepsilon = \mu^{1/3}$

$$\omega = n + \alpha^{\mathrm{cr}}\varepsilon^2 + \sum_{k=3}^{\infty}\alpha_k\varepsilon^k,\ x(t) = \sum_{k=1}^{\infty}x_k(t)\varepsilon^k,\ y(t) = \sum_{k=1}^{\infty}y_k(t)\varepsilon^k \tag{2.1}$$

Substituting expressions (2.1) into (1.1) and equating coefficients of like powers of ε, we arrive at the following systems of equations for the functions $x_k(t)$ and $y_k(t)(k = 1, 2, \ldots)$

$$dx_k / dt = -ny_k - \alpha_{k-1}y_1 + G_1^{(k)},\ dy_k / dt = nx_k + \alpha_{k-1}x_1 + G_2^{(k)} \tag{2.2}$$

where $G_j^{(k)}(j = 1, 2)$ are integral rational functions of $x_1, \ldots, x_{k-1}, y_1, \ldots, y_{k-1}, \alpha_2, \ldots, \alpha_{k-2}$ and are 2π-periodic in t.

The general solution of Eqs (2.2) has the form

$$x_k(t) = M_k^{(2)}\cos nt - M_k^{(1)}\sin nt + \varphi_k,\ y_k(t) = M_k^{(2)}\sin nt + M_k^{(1)}\cos nt + \psi_k \tag{2.3}$$

where $M_i^{(m)}$ $(i = 1, 2, \ldots; m = 1, 2)$ are arbitrary constants and φ_k and ψ_k are integral rational functions of $M_1^{(m)}, \ldots, M_{k-1}^{(m)}, \alpha_2, \ldots, \alpha_{k-2}$ and 2π-periodic of t.

We introduce the notation

$$I_k^{(1)} = \frac{1}{2\pi}\int_0^{2\pi}\{(-\alpha_{k-1}y_1 + G_1^{(k)})\cos nt + (\alpha_{k-1}x_1 + G_2^{(k)})\sin nt\}dt$$

$$I_k^{(2)} = \frac{1}{2\pi}\int_0^{2\pi}\{(-\alpha_{k-1}y_1 + G_1^{(k)})\sin nt - (\alpha_{k-1}x_1 + G_2^{(k)})\cos nt\}dt$$

The equations representing necessary and sufficient conditions for system (2.2) to have 2π-ps have the form

$$I_k^{(l)} = 0,\quad l = 1, 2 \tag{2.4}$$

Let us consider the nature of the dependence of $I_k^{(l)}$ on $M^{(m)}{}_i$. Since the expansions of $X(x, y)$ and $Y(x, y)$ in powers of x and y begin the terms of degree at least 2, it follows that when $k = 2q$ (where q is a natural number) $G_j^{(k)}$ is a linear function of $x_{q+1}, x_{q+2}, \ldots, x_{k-1}, y_{q+1}, y_{q+2}, \ldots, y_{k-1}$, but the quantities x_q and y_q occur in the function $G_j^{(k)}$ to second and first degree only. In the integrand of $I_k^{(l)}$, the coefficients of the second powers of $M_q^{(m)}$ and the first powers of $M_{k-1}^{(m)}$ are cubic polynomials in sin nt and cos nt, whose integrals from 0 to 2π always vanish. Thus, $I_k^{(l)}$ depends linearly on $M_q^{(m)}, M_{q+1}^{(m)}, \ldots, M_{k-2}^{(m)}$ and does not depend on $M_i^{(m)}$, $i > k - 2$. It can be shown similarly that for $k = 2q + 1$ $(q = 2, 3, \ldots)$ the integrals $I_k^{(l)}$ depend linearly on $M_{q+1}^{(m)}, M_{q+2}^{(m)}, \ldots, M_{k-2}^{(m)}$ and do not depend on $M_i^{(m)}$, $i > k - 2$. It can also be shown that for $k = 2q + 1(q = 2, 3, \ldots)$ the integrals $I_k^{(l)}$ contain only terms of second and first degree in $M_q^{(m)}$ $(q = 2, 3, \ldots)$.

We note further the following relations

$$\frac{\partial I_k^{(l)}}{\partial M_i^{(m)}} = \frac{\partial I_{k-p}^{(l)}}{\partial M_{i-p}^{(m)}}\ (k \geqslant i + 2;\ m = 1, 2;\ l = 1, 2) \tag{2.5}$$

$$\frac{\partial^2 I_k^{(l)}}{\partial M_i^{(m)}\partial M_j^{(n)}}=\frac{\partial^2 I_{k-p-s}^{(l)}}{\partial M_{i-p}^{(m)}\partial M_{j-s}^{(n)}}\quad (k\geqslant p+s+2;\ n=1,2;\ m=1,2;\ l=1,2)$$

which may be proved by mathematical induction.

We will now describe an algorithm for calculating the constants $M_i^{(m)}$, α_i. For $k = 1, 2$, Eqs (2.4) are identically true. For $k = 3$, Eqs (2.4), taking (1.2) and (1.4) into account, may be written explicitly as (2.6)

$$\chi M_1^{(m)}[(M_1^{(1)})^2+(M_1^{(2)})^2]-\alpha^{\rm cr}M_1^{(m)}-(-1)^m D_m=0,\ l \tag{2.6}$$

$$\chi=(3a_{03}-b_{12}+a_{21}-3b_{30})/8$$

It can be shown that $\chi = nh_2$, so that, by the results of Section 1, we have

$$\alpha^{\rm cr}=3[\chi(D_1^2+D_2^2)/4]^{1/3}$$

System (2.6) has exactly two solutions, one of which has the form

$$M_1^{(m)}=2(-1)^m D_m/[2\chi(D_1^2+D_2^2)]^{1/3},\quad m=1,2 \tag{2.7}$$

The quantities $M_1^{(m)}$, calculated from formulae (2.7), correspond to an isolated 2π-ps of system (1.1), which exists for any values of the parameter α, and for which the Jacobian of system (2.6) is always non-zero. Both on and off the bifurcation curve, this solution may be constructed by the method described in [1]. Following that method, the quantities $M_1^{(m)}$ $(i \geqslant 2)$ are uniquely determined from the system of equations (2.4) for $k = i + 2$. When that is done, the expressions for $M_i^{(m)}$ will depend on α_j $(i \leqslant j)$. The coefficients of the bifurcation curve α_j will be determined in the process of constructing the second 2π-ps of system (1.1), for which $M_i^{(m)}$ are determined from the formulae

$$M_1^{(m)}=(-1)^{m+1}D_m/[2\chi(D_1^2+D_2^2)]^{1/3},\quad m=1,2 \tag{2.8}$$

We will consider in detail a method for constructing this solution. Let Δ_1 denote the Jacobian of system (2.6). The determinant Δ_1 vanishes for the solution (2.8). As shown previously, system (1.1) has a unique 2π-ps on the bifurcation curve satisfying this condition. Therefore, to determine $M_i^{(m)}$ $(i \geqslant 2)$, we shall supplement the 2π-periodicity conditions (2.4) by the uniqueness condition, which may be satisfied by an appropriate choice of α_j.

For $k = 4$, the system of equations (2.4) may be written in the form

$$\frac{\partial I_4^{(l)}}{\partial M_2^{(1)}}M_2^{(1)}+\frac{\partial I_4^{(l)}}{\partial M_2^{(2)}}M_2^{(2)}-\alpha_3 M_1^{(l)}+\tilde I_4^{(l)}=0,\quad l=1,2 \tag{2.9}$$

This system is linear in $M_2^{(m)}$. The quantities $I_4^{-(l)}$ are defined if $M^{(m)}{}_i$ are known. Using relations (2.5), one can readily show that the determinant of the matrix of system (2.9) is equal to Δ_1 and therefore vanishes. The consistency condition for system (2.9) can always be satisfied by an appropriate choice of α_3. The equations of system (2.9) are linearly dependent, the second equation being obtained by multiplying the first by

$$-\gamma_1=\left(\frac{\partial I_3^{(2)}}{\partial M_1^{(1)}}\right)\left(\frac{\partial I_3^{(1)}}{\partial M_1^{(1)}}\right)^{-1}=\left(\frac{\partial I_3^{(2)}}{\partial M_1^{(2)}}\right)\left(\frac{\partial I_3^{(1)}}{\partial M_1^{(2)}}\right)^{-1} \tag{2.10}$$

This means that $M_2^{(m)}$ cannot be determined uniquely from system (2.9). Taking formulae (2.5) into account, we write the system of equations (2.4) for $k = 5$

$$\frac{\partial I_3^{(l)}}{\partial M_1^{(1)}}M_3^{(1)}+\frac{\partial I_3^{(l)}}{\partial M_1^{(2)}}M_3^{(2)}-\alpha_4 M_1^{(l)}+\tilde I_5^{(l)}(M_2^{(1)},M_2^{(2)})=0,\quad l=1,2 \tag{2.11}$$

where $\tilde I_5^{(l)}$ are polynomials of degree 2 in $M_2^{(m)}$.

System (2.11) is linear in $M_3^{(m)}$. The determinant of the matrix of this system is $\Delta_1 = 0$. This enables us to eliminate the unknowns $M_3^{(m)}$. Indeed, after multiplying the first equation by γ_1 and adding the result to the second, we obtain

$$-\alpha_4(\gamma_1 M_1^{(1)} + M_1^{(2)}) + \tilde{I}_5^{(2)}(M_2^{(1)}, M_2^{(2)}) + \gamma_1 \tilde{I}_5^{(1)}(M_2^{(1)}, M_2^{(2)}) = 0 \tag{2.12}$$

Equations (2.12) contain only terms of first and second degree in $M_2^{(m)}$. Together with the first equation of (2.9), Eq. (2.12) forms a system of equations in $M_2^{(m)}$ which has a unique solution, provided that its Jacobian, which we will denote henceforth by Δ_2, vanishes.

The equation $\Delta_2 = 0$, together with the first equation of (2.9), forms a system of linear equations in $M_2^{(m)}$. The determinant of the matrix of this system, calculated using (2.5), is

$$\Delta = 24\chi^3 M_1^{(2)}[(M_1^{(1)})^2 + (M_1^{(m)})^2]^2 \tag{2.13}$$

and therefore does not vanish. Hence the system has a unique solution. After $M_2^{(m)}$ has been determined from Eq. (2.12), α_4 is uniquely determined. We also observe that Eq. (2.12) is the condition for consistency of the system of equations (2.11), which is linear in $M_3^{(m)}$.

To determine $M_3^{(m)}$ uniquely, one must use the 2π-periodicity conditions for the system of linear equations for $k > 5$.

We introduce the notation

$$\gamma_2 = -\left(\gamma_1 \frac{\partial I_5^{(1)}}{\partial M_2^{(1)}} + \frac{\partial I_5^{(2)}}{\partial M_2^{(1)}}\right)\left(\frac{\partial I_3^{(1)}}{\partial M_1^{(1)}}\right)^{-1} \tag{2.14}$$

It then follows from the equations $\Delta_2 = 0$ and from (2.14) that

$$\frac{\partial I_5^{(2)}}{\partial M_2^{(m)}} + \gamma_1 \frac{\partial I_5^{(1)}}{\partial M_2^{(m)}} + \gamma_2 \frac{\partial I_3^{(1)}}{\partial M_1^{(m)}} = 0, \quad m = 1,2 \tag{2.15}$$

In view of relations (2.5), Eqs (2.4) for $k = 6$ become

$$\frac{\partial I_3^{(l)}}{\partial M_1^{(1)}} M_4^{(1)} + \frac{\partial I_3^{(l)}}{\partial M_1^{(2)}} M_4^{(2)} + \frac{\partial I_5^{(l)}}{\partial M_2^{(1)}} M_3^{(1)} + \frac{\partial I_5^{(l)}}{\partial M_2^{(2)}} M_3^{(2)} - \alpha_5 M_1^{(l)} + \tilde{I}_6^{(l)} = 0, \quad l = 1,2 \tag{2.16}$$

Let us consider a linear combination of equations (2.11) and (2.16): $I_6^{(2)} + \gamma_1 I_6^{(1)} + \gamma_2 I_5^{(1)} = 0$, which, using (2.10) and (2.15), we transform into

$$-\alpha_5(M_1^{(2)} + \gamma_1 N_1^{(1)}) - \alpha_4 M_1^{(2)}\gamma_2 + \tilde{I}_6^{(2)} + \gamma_1 \tilde{I}_6^{(1)} + \gamma_2 \tilde{I}_5^{(1)} = 0 \tag{2.17}$$

Equation (2.17) does not depend on $M_3^{(m)}$ and $M_4^{(m)}$, and it may be satisfied by suitable choice of α_5. Note that condition (2.17) itself guarantees the consistency of the system of equations (2.11), (2.16), which is linear in $M_3^{(m)}$ and $M_4{}^{(m)}$.

One could try to eliminate $M^{(m)}{}_4$ from (2.16) through the equation $I_6^{(2)} + \gamma_1 I_6^{(1)} = 0$ (which does not contain $M^{(m)}{}_4$), but, by (2.17), the equation thus obtained will be linearly dependent on Eqs (2.11). Thus, unique determination of $M_3^{(m)}$ requires the use of Eqs (2.4) for $k = 7$

$$\begin{aligned}&\frac{\partial I_3^{(l)}}{\partial M_1^{(1)}} M_5^{(1)} + \frac{\partial I_3^{(l)}}{\partial M_1^{(2)}} M_5^{(2)} + \frac{\partial I_5^{(l)}}{\partial M_2^{(1)}} M_4^{(1)} + \frac{\partial I_5^{(l)}}{\partial M_2^{(2)}} M_4^{(2)} - \alpha_6 M_1^{(l)} + \\ &+ \tilde{I}_7^{(l)}(M_3^{(1)}, M_3^{(2)}) = 0, \quad l = 1,2\end{aligned} \tag{2.18}$$

where $I_7^{(l)}(M_3^{(1)}, M_3^{(2)})$ are polynomials of the second degree in $M_3^{(m)}$.

Proceeding as for (2.17), we obtain from Eqs (2.16) and (2.18) an equation $I_7^{(2)} + \gamma_1 I_7^{(1)} + \gamma_2 I_6^{(1)} = 0$ that does not contain the unknowns $M_3^{(m)}$ and $M_4^{(m)}$

$$\gamma_2\left(\frac{\partial I_5^{(1)}}{\partial M_2^{(2)}} M_3^{(2)} + \frac{\partial I_5^{(1)}}{\partial M_2^{(1)}} M_3^{(1)}\right) - \alpha_6(M_1^{(2)} + \gamma_1 M_1^{(1)}) - \gamma_2 \alpha_5 M_1^{(1)} + \tilde{I}_7^{(2)}(M_3^{(1)}, M_3^{(2)}) + \tag{2.19}$$

$$+\gamma_1\tilde{I}_7^{(1)}(M_3^{(1)},M_3^{(2)})+\gamma_2\tilde{I}_6^{(2)}=0$$

Equation (2.19) depends only on the unknowns $M_3^{(m)}$ and the parameters α_6; together with the first equation of (2.11), it forms a system of equations in $M_3^{(m)}$ which has a unique solution, provided that its Jacobian Δ_3 vanishes. The first equation of (2.11) and the equation $\Delta_3 = 0$ form a system of equations which is linear in $M_3^{(m)}$. The determinant of its matrix is equal to Δ (2.13) and consequently does not vanish. This means that the system has a unique solution. After $M_3^{(m)}$ has been determined from Eq. (2.19), α_6 is uniquely defined.

Let us suppose, then, that for $k = 2p - 1$ we have already found $M_1^{(m)},\ldots,M_{p-1}^{(m)}$, $\alpha_2,\ldots,\alpha_{2p-2}$, and the following equations from (2.4) have been written out

$$\begin{gathered}
\text{for } k=p+2\\
\frac{\partial I_3^{(l)}}{\partial M_1^{(1)}}M_p^{(1)}+\frac{\partial I_3^{(l)}}{\partial M_1^{(2)}}M_p^{(2)}-\alpha_{p+1}M_1^{(l)}+\tilde{I}_{p+2}^{(l)}=0\\
\text{for } k=p+3\\
\frac{\partial I_3^{(l)}}{\partial M_1^{(1)}}M_{p+1}^{(1)}+\frac{\partial I_3^{(l)}}{\partial M_1^{(2)}}M_{p+1}^{(2)}+\frac{\partial I_5^{(l)}}{\partial M_2^{(1)}}M_p^{(1)}+\frac{\partial I_5^{(l)}}{\partial M_2^{(2)}}M_p^{(2)}-\alpha_{p+2}M_1^{(l)}+\tilde{I}_{p+3}^{(l)}=0\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\text{for } k=2p\\
\frac{\partial I_3^{(l)}}{\partial M_1^{(1)}}M_{2p-2}^{(1)}+\frac{\partial I_3^{(l)}}{\partial M_1^{(2)}}M_{2p-2}^{(2)}+\frac{\partial I_5^{(l)}}{\partial M_2^{(1)}}M_{2p-3}^{(1)}+\frac{\partial I_5^{(l)}}{\partial M_2^{(2)}}M_{2p-3}^{(2)}+\ldots\\
\ldots+\frac{\partial I_{2p}^{(l)}}{\partial M_p^{(1)}}M_p^{(1)}+\frac{\partial I_{2p}^{(l)}}{\partial M_p^{(2)}}M_p^{(2)}-\alpha_{2p-1}M_1^{(l)}+\tilde{I}_{2p}^{(l)}=0,\quad l=1.2
\end{gathered}\tag{2.20}$$

These are all equations in the unknowns $M_p^{(m)},\ldots,M_{2p-2}^{(m)}$. In addition, the conditions for the solution to be unique imply the following relations

$$\begin{gathered}
\text{for } k=3\\
\frac{\partial I_3^{(2)}}{\partial M_1^{(m)}}+\gamma_1\frac{\partial I_3^{(1)}}{\partial M_1^{(m)}}=0\\
\text{for } k=5\\
\frac{\partial I_5^{(2)}}{\partial M_2^{(m)}}+\gamma_1\frac{\partial I_5^{(1)}}{\partial M_2^{(m)}}+\gamma_2\frac{\partial I_3^{(1)}}{\partial M_1^{(m)}}=0\\
\cdots\cdots\cdots\cdots\cdots\\
\text{for } k=2p-1\\
\frac{\partial I_{2p-1}^{(2)}}{\partial M_{p-1}^{(m)}}+\gamma_1\frac{\partial I_{2p-1}^{(1)}}{\partial M_{p-1}^{(m)}}+\gamma_2\frac{\partial I_{2p-3}^{(1)}}{\partial M_{p-2}^{(m)}}+\ldots+\gamma_{p-1}\frac{\partial I_3^{(1)}}{\partial M_1^{(m)}}=0,\quad m=1,2
\end{gathered}\tag{2.21}$$

The quantities $\gamma_1,\ldots,\gamma_{p-1}$ have already been calculated, by formulae analogous to (2.10) and (2.14).

In Eqs (2.20), apart from the unknowns $M_p^{(m)},\ldots,M_{k-2}^{(m)}$, there is still an undetermined quantity α_{2p-1}. Relations (2.21) enable us, after eliminating the unknowns $M_i^{(m)}$ ($i = p,\ldots,k+2$) from Eqs (2.21), to write down an equation for α_{2p-1}, which may be obtained as a linear combination of Eqs (2.20)

$$I_{2p}^{(2)}+\gamma_1 I_{2p}^{(1)}+\gamma_2 I_{2p-1}^{(1)}+\gamma_3 I_{2p-2}^{(1)}+\ldots+\gamma_{p-1}I_{p+2}^{(1)}=0$$

To determine $M_p^{(m)}$, we add Eqs (2.4) for $k = 2p + 1$ to Eqs (2.20)

$$\frac{\partial I_3^{(l)}}{\partial M_1^{(1)}}M_{2p-1}^{(1)}+\frac{\partial I_3^{(l)}}{\partial M_1^{(2)}}M_{2p-1}^{(2)}+\frac{\partial I_5^{(l)}}{\partial M_2^{(1)}}M_{2p-2}^{(1)}+\frac{\partial I_5^{(l)}}{\partial M_2^{(2)}}M_{2p-2}^{(2)}+\ldots$$

$$\ldots+\frac{\partial I_{2p+1}^{(l)}}{\partial M_p^{(1)}}M_p^{(1)}+\frac{\partial I_{2p+1}^{(l)}}{\partial M_p^{(2)}}M_p^{(2)}-\alpha_{2p}M_1^{(l)}+\tilde{I}_{2p+1}^{(l)}(M_p^{(1)},M_p^{(2)})=0,\quad l=1,2 \tag{2.22}$$

A linear combination of Eqs (2.20) and (2.22)

$$I_{2p+1}^{(2)}+\gamma_1 I_{2p+1}^{(1)}+\gamma_2 I_{2p}^{(1)}+\gamma_3 I_{2p-1}^{(1)}+\ldots+\gamma_{p-1}I_{p+3}^{(1)}=0 \tag{2.23}$$

contains only the unknowns $M_p^{(m)}$, which occur in the first and second powers, as well as the undetermined quantity α_{2p}. The first equation of (2.20) and Eq. (2.23) form a system of equations in $M_p^{(m)}$. The uniqueness condition for the solutions of this system, together with the first equation of (2.20), form a system of equations which is linear in $M_p^{(m)}$. The determinant of the matrix of this system equals Δ (2.13) and consequently does not vanish. Thus, the system has a unique solution. Once the $M_p^{(m)}$ have been found, the quantity α_{2p} is determined from Eq. (2.23).

3. EXAMPLES

The results of Sections 1 and 2 can be used to investigate bifurcation of the motions of mechanical systems of quite a wide class. We will consider two examples.

A pendulum with vibrating suspension point. Let x be the angle by which a pendulum of length l, whose suspension point is vibrating horizontally with amplitude a and frequency Ω, deviates from the vertical. The dissipative forces acting on the pendulum are given by the Rayleigh function $R=\chi\dot{x}^2/2$ (where the dot, both here and later, stands for differentiation with respect to dimensionless time $\tau=\Omega t$). The equation of motion for the pendulum is

$$\ddot{x}+\chi\dot{x}+\omega_0^2\sin x=\varepsilon\sin\tau\cos x;\quad \omega_0^2=g/(\Omega^2 l),\quad \varepsilon=a/l \tag{3.1}$$

Equation (3.1) may be replaced by an equivalent system of two equations

$$\dot{y}=-\omega_0\sin x+(\varepsilon/\omega_0)\sin\tau\cos x-\chi y,\quad \dot{x}=\omega_0 y \tag{3.2}$$

Let us assume that ε and χ are small quantities of the same order, i.e., $\chi=b\varepsilon\ll 1$, $b=0(1)$. Under other assumptions as to the orders of smallness of ε and χ, the problem of the bifurcation of 2π-ps of this pendulum has been studied before [5].

System (3.2) is a system of type (1.1). Therefore, using known results [2] and those of Section 2, if ω_0 is close to unity, one has bifurcation of the 2π-periodic oscillations of the pendulum which, when $\varepsilon=0$, become a stable equilibrium position ($x=0$). The bifurcation curve and the 2π-ps on it, constructed by the method presented in Section 2, are

$$\omega_0=1+\varepsilon^{2/3}\cdot 3\cdot 2^{1/3}/8-\varepsilon^{4/3}(45+64b^2)/256+O(\varepsilon^2) \tag{3.3}$$

$$x_1=\varepsilon^{1/3}\sin\tau-\varepsilon^{2/3}b\cdot 2^{4/3}\cos\tau+\varepsilon((6-64b^2)\sin\tau+\sin 3\tau)/48+O(\varepsilon^{4/3}) \tag{3.4}$$

$$x_2=-\varepsilon^{1/3}\cdot 2\cdot 2^{2/3}\sin\tau-\varepsilon^{2/3}8b\cos\tau-\varepsilon((57+448b^2)\sin\tau-4\sin 3\tau)/24+O(\varepsilon^{4/3}) \tag{3.5}$$

On crossing the bifurcation curve, the 2π-ps (3.4) either disappears or generates two new 2π-ps.

Oscillations of an artificial satellite in the plane of a weakly elliptical orbit. The planar motions of an artificial satellite, considered as a rigid body, about its centre of mass in a central Newtonian gravitational field in an elliptical orbit are described by the following equation [6]

$$(1+e\cos\nu)d^2\psi/d\nu^2-2e\sin\nu\, d\psi/d\nu+\omega_0^2\sin\psi\cos\psi=2e\sin\nu \tag{3.6}$$

where ψ is the angle between one of the principal central axes of inertia of the satellite in the orbital plane and the radius vector of its centre of mass, e is the orbit eccentricity and ν is the true anomaly; $\omega_0^2=3(A-C)/B$, where A, B and C are the moments of inertia of the satellite about its principal central axes. In the case of a circular orbit ($e=0$), Eq. (3.6) has a particular solution $\psi=0$, corresponding to an equilibrium position of the satellite in an orbital system of coordinates. If $e\ll 1$, the equilibrium $\psi=0$ gives way to oscillations, which are odd functions of ν and 2π-periodic. If $\omega_0\approx 1$, these solutions bifurcate [6]. Other methods [6–8] have yielded an approximate expression for the bifurcation curve. Using the technique of Section 2, we will derive a more accurate expression for the bifurcation curve. To that end, we replace Eq. (3.6) by an equivalent Hamiltonian equation system

$$dq/d\nu = \partial H/\partial p, \quad dp/d\nu = -\partial H/\partial q \tag{3.7}$$

with Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{e\cos\nu}{2(1+e\cos\nu)}q^2 + \frac{1}{2}\omega_0^2(1+e\cos\nu)\sin^2\left(\frac{q}{1+e\cos\nu}\right) - 2eq\sin\nu \tag{3.8}$$

The canonically conjugate coordinate q and momentum p are introduced by the formulae

$$\psi = \frac{q}{1+e\cos\nu}, \quad p = \frac{dq}{d\nu} \tag{3.9}$$

The canonical change of variables $q_* = \sqrt{(\omega_0)}q$, $p_* = p/\sqrt{(\omega_0)}$ reduces system (3.7) to the form of (1.1). The bifurcation curve and 2π-ps on it, found by the method of Section 2, are

$$\omega_0 = 1 + e^{2/3}\cdot 3\cdot 2^{2/3}/4 + e^{4/3}\cdot 3\cdot 2^{1/3}/32 - e^2\cdot 49/256 + O(e^{8/3}) \tag{3.10}$$

$$\begin{aligned}\psi_1(\nu) &= e^{1/3}\cdot 2^{1/3}\sin\nu - e(18\sin\nu - \sin 3\nu)/24 - e^{4/3}\cdot 2^{-2/3}\sin 2\nu + \\ &+ e^{5/3}\cdot 2^{2/3}(190\sin\nu + 5\sin 3\nu + \sin 5\nu)/640 + e^2(45\sin 2\nu - 13\sin 4\nu)/240 + O(e^{7/3})\end{aligned} \tag{3.11}$$

$$\begin{aligned}\psi_2(\nu) &= e^{1/3}\cdot 2^{4/3}\sin\nu + e(9\sin\nu - 4\sin 3\nu)/12 + e^{4/3}\cdot 2^{1/3}\sin 2\nu + \\ &+ e^{5/3}\cdot 2^{2/3}(5\sin\nu + 5\sin 3\nu - 2\sin 5\nu)/40 + e^2(225\sin 2\nu - 52\sin 4\nu)/120 + O(e^{7/3})\end{aligned} \tag{3.12}$$

On crossing the bifurcation curve, solution (3.11) either disappears or generates two new 2π-ps.

To construct expansions (3.3)–(3.5), (3.10)–(3.12), we have written a program implementing the algorithm of Section 2 in the MAPLE analytical transformation system.

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